Last Time: $\text{Curl}(\vec{v}) = \nabla \times \vec{v}$ $\Big\}$ $\text{div}(\vec{v}) = \nabla \cdot \vec{v}$

$\vec{v} = (P, Q, R)$

Prop: ① $\text{Curl}(\nabla f) = \vec{0}$ and ② $\text{div}(\text{curl}(\vec{v})) = 0$

Note: ① Intuitively the divergence of a vector field calculates "how badly does the v.f. want to leave a bounded set"

② The curl is a measure of "how swirly" a v.f. wants to be

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Recasting Green's Theorem

Let $\vec{v} = \langle P, Q, 0 \rangle$ have cts. partial derivatives on an open region $R$ containing $D$, where $D$ is a closed region with a piecewise smooth boundary curve

Then $\iint_D \text{curl}(\vec{v}) \cdot \vec{k} \, dA = \int_{\partial D} \vec{v} \cdot d\vec{r}$

$$\int_{\partial D} \vec{v} \cdot (y'(t)\vec{i} - x'(t)\vec{j}) \cdot \frac{1}{|\vec{r}(t)|} \cdot ds = \iint_D \text{div}(\vec{v}) \, dA$$

$$\text{curl}(\vec{v}) = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} = \langle -Q_z, P_z, Q_x - P_y \rangle$$

$$\therefore \text{curl}(\vec{v}) \cdot \vec{k} = Q_x - P_y = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\iint_D \text{curl}(\vec{v}) \cdot \vec{k} \, dA = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} \vec{v} \cdot d\vec{r}$$

Green's Theorem

$$\operatorname{div}(\vec{v}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, 0 \rangle$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

$$\therefore \iint_{\partial D} \operatorname{div}(\vec{v}) \cdot dA = \iint \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA \quad \rightsquigarrow \vec{w} = \langle -Q, P, 0 \rangle$$

Greens Theorem $\rightarrow$ $$= \int_{\partial D} \vec{w} \cdot d\vec{r} = \int_a^b (-Q x'(t) + P y'(t)) dt$$

$$= \int_a^b \langle P, Q \rangle \cdot \langle y', x' \rangle \, dt$$ $\langle x, y \rangle$ an arc-length

$$= \int_{\partial D} \vec{v} \cdot (y'(t)\vec{i} - x'(t)\vec{j}) \frac{1}{|r'(t)|} ds$$

Point! Greens Theorem can be recast using

1. Curl
2. Divergence

These two ways of recasting Green's Theorem lead to two seperate generalizations of Green's Theorem

1. Stokes's Theorem
2. Divergence Theorem

## §16.6 Parametric Surfaces (Not on Exam 3)

Def: A parametric surface is a function
$\vec{S}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$
for some domain in $\mathbb{R}^2$

Idea: This is a "space curve in dimension 2"

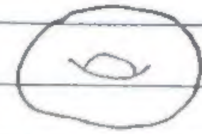
Ex A sphere of radius $r > 0$ can be parameterized as: $\vec{S}(\theta,\varphi) = \langle r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi)\rangle$

↑ from spherical coords.

on $D = [0,2\pi] \times [0,\pi]$

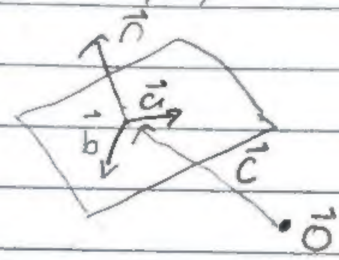
Ex The torus has parameterization
$\vec{S}(u,v) = \langle (2+\sin(v))\cos(u), (2+\sin(v))\sin(u), \cos(v)\rangle$
On $D = [0,2\pi] \times [0,\pi]$

Torus →



Ex every plane can be parametrized via

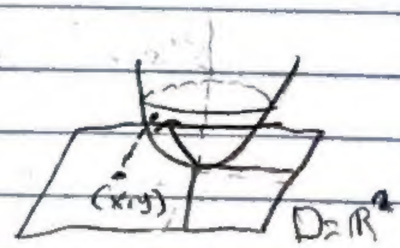
$\vec{S}(u,v) = u\vec{a} + v\vec{b} + \vec{c}$ for suitable $\vec{a}, \vec{b}, \vec{c}$
for $D = \mathbb{R}^2$




Idea: $\Pi$ is just determined by points $(u,v)$ in $\mathbb{R}^2$ via $\vec{a}, \vec{b}, \vec{c}$ and the above equation

Ex: Compute a parameterization for the paraboloid $z = x^2 + 2y^2$

Note: there are many ways to do this




Sol①: $\vec{S}(x,y) = \langle x, y, x^2 + 2y^2\rangle$

Sol②: $\vec{S}(r,\theta) = \langle r\cos\theta, r\sin\theta, (r\cos\theta)^2 + (2r\sin\theta)^2\rangle$
$D = [0,\infty] \times [0,2\pi]$ $= \langle r\cos\theta, r\sin\theta, r^2(1+\sin^2\theta)\rangle$

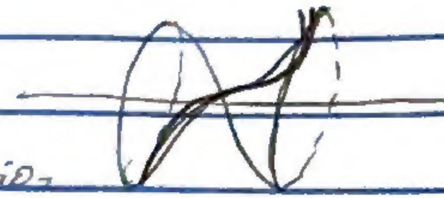
Sol③: $\vec{S}(r,\theta) = \langle \sqrt{2}\, r\cos\theta, r\sin\theta, 2r^2\rangle$

Ex Let $f(t)$ be a single variable function. The surface of revolution obtained by revolving $f$ about the $x$-axis is parameterized by

$$\vec{s}(x,\theta) = \langle x, f(x)\cos(\theta), f(x)\sin(\theta)\rangle$$

Sub-ex: Let $f(x) = x^3$

the surface has parameterization



$$\vec{s}(x,\theta) = \langle x, x^3\cos(\theta), x^3\sin(\theta)\rangle$$